Thm) Rellich - Kondrachov Compactness.

Let $\{u_k\} \subseteq H_0^1(\Omega)$ $(\subset L^2(\Omega))$ satisfy

$$\|u_k\|_{H^1} \leq C \quad \forall k \in \mathbb{N}.$$

where C is independent of k.

Then $\exists$ a subseq $\{u_{k_j}\} \subset \{u_k\}$

and $u \in L^2(\Omega)$ s.t.

$$\lim_{k_j \to +\infty} \|u_{k_j} - u\|_{L^2} = 0.$$

i.e. $u_{k_j} \longrightarrow u$ in L^2

(Ω is bounded and $\partial\Omega \in C^1$)

Remark) $\{u_k\} \subset W^{1,p}$, $\|u_k\|_{W^{1,p}} \le C$.

Let $0 < q$ satisfy

$$q \le \frac{np}{n-p} \quad \text{if } p < n,$$

$$q < +\infty \quad \text{in } p \ge n.$$

Then $\exists$ a subseq $\{u_{k_j}\} \subseteq \{u_k\}$ such that u_{k_j} is convergent (strongly) in $L_q(\Omega)$.

Application 1) Existence of the 1st eigenfunction w_1 in H_0^1 and existence of w_j.

Appl 2) Divergence λ_j

$$\lim_{\lambda_j \to +\infty} \lambda_j = +\infty$$

Def) positive symmetric mollifier $\eta : \mathbb{R}^n \to \mathbb{R}$

$\eta(x) \geq 0$, $\eta(x) = 0$ if $|x| \geq 1$.

$\int_{\mathbb{R}^n} \eta = 1$. $\eta(x) = \psi(|x|)$. $\eta \in C^\infty$

Given $\varepsilon > 0$, $\eta_\varepsilon(x) = \varepsilon^{-n} \eta(x/\varepsilon)$

$(\Rightarrow \eta_\varepsilon(x) = 0$ if $|x| \geq \varepsilon)$

Def) Convolution of $f, g : \mathbb{R}^n \to \mathbb{R}$.

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x-y) \, dy$$

Remark) Given any $f \in L^1_{loc}(\mathbb{R}^n)$

$$f * \eta_\varepsilon(x) = \int_{\mathbb{R}^n} f(y) \eta_\varepsilon(x-y) \, dy \in C^\infty$$

$(\varepsilon z = y - x)$

$$= \int_{\mathbb{R}^n} f(x+\varepsilon z) \eta_\varepsilon(\varepsilon z) \varepsilon^n \, dz$$

$$= \int_{B_1(0)} f(x+\varepsilon z) \eta(z) \, dz \quad \text{—} \; ①$$

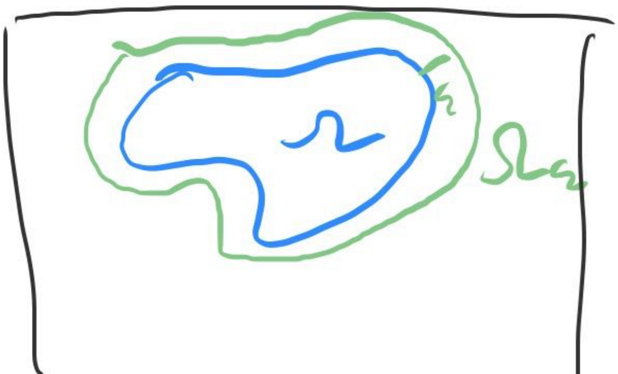
prop 1) Given $u_m \in H_0^1(\Omega)$, $\varepsilon > 0$

$$U_m^\varepsilon \triangleq \bar{u}_m * \eta_\varepsilon \in C^\infty$$

where $\bar{u}_m = \begin{cases} u_m & \text{in } \Omega \\ 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$

Then. $u_m^\varepsilon = 0$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$

where $\Omega_\varepsilon = \{ x \in \mathbb{R}^n \mid d(x, \Omega) < \varepsilon \}$.



Moreover.

$$\lim_{\varepsilon \to 0} \| u_m^\varepsilon - \bar{u}_m \|_{L^2(\mathbb{R}^n)} = 0$$

proof) $u_m^\varepsilon(x) = \int_{B_1(0)} \bar{u}_m(x + \varepsilon z) \eta(z) \, dz$

If $d(x, \Omega) \geq \varepsilon$, then $x + \varepsilon z \notin \Omega$ $\forall z \in B_1(0)$

(i.e. $\bar{u}_m(x + \varepsilon z) = 0$)

$\Rightarrow$ $u_m^\varepsilon(x) = 0$ in $\mathbb{R}^n \setminus \Omega_\varepsilon$.

Recall $\int_{B_1(0)} \eta(z)\,dz = 1$

$$\Rightarrow \bar{u}_m(x) = \int_{B_1(0)} \bar{u}_m(x)\,\eta(z)\,dz.$$

By ①.

$$u_m^\varepsilon(x) - \bar{u}_m(x)$$

$$= \int_{B_1} \eta(z)\left(\bar{u}_m(x+\varepsilon z) - \bar{u}_m(x)\right)dz.$$

$$= \int_{B_1} \eta(z) \int_0^1 \frac{d}{dt}\bar{u}_m(x+\varepsilon t z)\,dt\,dy.$$

$(\because$ if $u_m \in C_0^\infty(\Omega))$

$$= \varepsilon \int_{B_1} \eta(z) \int_0^1 \langle z, \nabla u_m(x+\varepsilon t z)\rangle\,dt\,dz$$

$$\Rightarrow \int_{\Omega_\varepsilon} |u_m^\varepsilon(x) - \bar{u}_m(x)|\, dx$$

$(|z| \leq 1)$

$$\leq \varepsilon \int_{B_1} \eta(z) \int_0^1 \int_{\Omega_\varepsilon} |\nabla \bar{u}_m(x + \varepsilon t z)|\, dx\, dt\, dz \quad — ②$$

Given $u_m \in H_0^1(\Omega)$. we have $\{u_{m,j}\} \subseteq C_0^\infty$

s.t. $\lim_{j \to +\infty} \|u_{m,j} - u_m\|_{H^1} = 0 \quad — ③$

we obtain ② for each $u_{m,j}$

$$\Rightarrow \|u_{m,j}^\varepsilon - \bar{u}_{m,j}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon \|\nabla \bar{u}_{m,j}\|_{L^1(\Omega_\varepsilon)}$$

$\bar{u}_{m,j} \to \bar{u}_m$ in $L^1(\Omega_\varepsilon)$

by ③, Hölder

$\nabla \bar{u}_{m,j} \to \nabla \bar{u}_m$ in $L^1(\Omega_\varepsilon)$

$$u^{\varepsilon}_{m,j}(x) - u^{\varepsilon}_{m}(x)$$

$$= \int_{B_1(0)} \eta(z)\left(\bar{u}_{m,j}(x+\varepsilon z) - \bar{u}_{m}(x+\varepsilon z)\right)$$

$$\int_{\Omega_\varepsilon} |u^{\varepsilon}_{m,j} - u^{\varepsilon}_{m}|$$

$$\leq \int_{B_1} \eta(z) \int_{\Omega_\varepsilon} |\bar{u}_{m,j}(x+\varepsilon z) - \bar{u}_{m}(x+\varepsilon z)|\, dx\, dz$$

$$= \|\bar{u}_{m,j} - \bar{u}_{m}\|_{L^1(\Omega_2)} \overbrace{\int_{B_1} \eta(z)\, dz}^{1}$$

$$\leq C(n,\Omega)\, \|u_{m,j} - u_m\|_{L^2(\Omega)} \to 0$$

Therefore. $\|u^{\varepsilon}_{m} - \bar{u}_{m}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon \|\nabla \bar{u}_m\|_{L^1}$

$$\therefore \|u^{\varepsilon}_{m} - \bar{u}_{m}\|_{L^1(\Omega_\varepsilon)} \leq C\varepsilon \|\nabla \bar{u}_m\|_{L^2}$$

$$\leq C\varepsilon \|\bar{u}_m\|_{H^1(\Omega_\varepsilon)} \quad \text{— } 4$$

On the other hand.

$$u_m^\varepsilon - \bar{u}_m \in H_0^1(\Omega_\varepsilon)$$

By Sobolev

$$\|u_m^\varepsilon - \bar{u}_m\|_{L^{p^*}(\Omega_\varepsilon)} \leq C\|\nabla u_m^\varepsilon - \nabla \bar{u}_m\|_{L^2}$$

where $p^* \leq \frac{2n}{n-2}$ if $n > 3$.

and $p^* < +\infty$ if $n = 2$

choose $p^* > 2$. Then Hölder implies

So called interpolation ineq.

$$\|u_m^\varepsilon - \bar{u}_m\|_{L^2(\Omega_\varepsilon)}$$

$$\leq \|u_m^\varepsilon - \bar{u}_m\|_{L^1}^{\theta} \|u_m^\varepsilon - \bar{u}_m\|_{L^{p^*}}^{1-\theta}$$

where $\frac{1}{2} = \theta + \frac{1-\theta}{p^*}$. $0 < \theta < 1$.

$$\left(\because \int u^{2\theta} u^{2(1-\theta)} \leq \left(\int |u|\right)^{2\theta} \left(\int |u|^{p^*}\right)^{\frac{2(1-\theta)}{p^*}} \right)$$

$\therefore \|u_m^\varepsilon - \bar{u}_m\|_{L^2} \leq C\varepsilon \to 0$ as $\varepsilon \to 0$ ▨

prop 2) For each $\varepsilon > 0$

u_m^ε are uniformly bounded

and equicontinous.

(i.e. we can apply Arzela-Ascoli)

proof) $|u_m^\varepsilon(x)| \le \int |\bar{u}_m(y)| \eta_\varepsilon(x-y) dy$

$$\le \left(\sup_{\mathbb{R}^n} \eta_\varepsilon\right) \|\bar{u}_m\|_{L^1(\Omega_\varepsilon)}$$

$$\le C \varepsilon^{-n} \|u_m\|_{L^2(\Omega)}$$

uniformly bounded. $\rightarrow$ $\le C \varepsilon^{-n}. \quad \forall x \in \mathbb{R}^n.$

$$|\nabla u_m^\varepsilon(x)| \le \int |\nabla \eta_\varepsilon(x-y)| |\bar{u}_m(y)| dy$$

$$\le C \varepsilon^{-n-1}$$

$\Rightarrow u_m^\varepsilon$ is equicontinuous

Lemma) Given ε,

$\{u_n\}$ has a subsequence

$\{u_{m_j}\}$ s.t. u_{m_j} converges uniformly.

i.e. $\lim_{\min\{m_j, m_k\} \to +\infty} \left(\sup_{\mathbb{R}^n} |u_{m_j} - u_{m_k}| \right) = 0$

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pf) prop 2 + Arzela - Ascoli!!

proof of RK theorem)

By prop 1) $\exists\ \varepsilon_1 > 0$ s.t.

$$\| u_m^{\varepsilon_1} - \bar{u}_m \|_{L^2} \leq 1/2 .$$

By Lemma, $\{u_m\}$ has a subseq. $u_{1,m}$ s.t.

$$\lim_{\min(m,k)\to+\infty} \sup_{\mathbb{R}^n} | u_{1,m}^{\varepsilon_1} - u_{1,k}^{\varepsilon_1} | = 0 .$$

$$\Rightarrow \lim_{m,k\to+\infty} \| u_{1,m}^{\varepsilon} - u_{1,k}^{\bar{\varepsilon}} \|_{L^2(\Omega_{\varepsilon_1})} = 0$$

$$\therefore \limsup_{m,k\to+\infty} \| \bar{u}_{1,m} - \bar{u}_{1,k} \|_{L^2(\Omega_{\varepsilon_1})} \leq 1$$

Next. $\exists\, \varepsilon_2 \in (0, \varepsilon_1)$ s.t.

$$\| u_m^{\varepsilon_2} - \bar{u}_m \| \leq 1/24.$$

$\{u_{1,m}\}$ has a subseq. $\{u_{2,m}\}$

s.t. $\lim\limits_{m,k} \sup\limits_{\varphi \geq 1} | u_{2,m}^{\varepsilon_2} - u_{2,k}^{\varepsilon_2} | = 0$

$$\Rightarrow \lim \| u_{2,m}^{\varepsilon_2} - u_{2,k}^{\varepsilon_2} \|_{L^2(\Omega_{\varepsilon_2})} = 0$$

$$\limsup_{m,k} \| \bar{u}_{2,m} - \bar{u}_{2,k} \|_{L^2(\Omega_{\varepsilon_2})} \leq \frac{1}{2}.$$

Similarly, $\{u_{2,m}\}$ has a subseq

$\{u_{3,m}\}$ s.t. $\limsup \| \bar{u}_{3,m} - \bar{u}_{3,k} \|_{L^2} \leq 1/4$

We iterate this process, and choose $\hat{u}_m = u_{m,m}$.

Then $\{\hat{u}_m\}$ is a subseq of $\{u_m\}$

and $\limsup_{m,k\to+\infty} \|\hat{u}_m - \hat{u}_k\|_{L^2(\Omega)} = 0$

namely, $\{\hat{u}_m\}$ is a cauchy seq in $L^2(\Omega)$, so $\exists\, u \in L^2(\Omega)$ s.t.

$\lim_{m\to+\infty} \|\hat{u}_m - u\|_{L^2} = 0$.

Cor 1) $\exists$ 1st egienfunction w_1 in H^1_0

See. lecture 10 (c).

Cor 2) The egienvalues λ_i diverges.

pf). Suppose $\lambda_i \le M$.

$$\|w_j\|_{H^1}^2 = \int_\Omega |\nabla w_j|^2 + w_j^2$$

$$= \int_\Omega -w_j \Delta w_j \, dx + 1$$

$$= \int_\Omega (V + \lambda_j) w_j^2 + 1$$

$$\le (M + \sup V) \|w_j\|_{L^2}^2 + 1$$

$$= M + 1 + \sup V = C$$

By the RK theorem.

$\exists$ a subseq w_{m_j} and $\bar{w} \in L^2$

s.t. $w_{m_j} \to \bar{w}$ in L^2. — ①

However, $\langle w_{m_j}, w_{m_l} \rangle_{L^2} = 0$

$\forall\, m_j \neq m_l$. — ②

① implies $\lim \| w_{m_j} - w_{m_l} \|_{L^2}^2 = 0$

② "

$$
\begin{aligned}
&\| w_{m_j} - w_{m_l} \|_{L^2}^2 \\
&= \overset{1}{\| w_{m_j} \|_{L^2}^2} - 2 \overset{0}{\langle w_{m_j}, w_{m_l} \rangle_{L^2}} \\
&\qquad + \| w_{m_l} \|_{L^2}^2 \ \overset{}{1} \\
&= 2 \neq 0.
\end{aligned}
$$

Ex) In $n=1$. $\Omega = (0, \pi)$

$$w_j = \sqrt{\frac{2}{\pi}} \sin jx$$

$$\left(\because \|w_j\|^2_{L^2} = \frac{2}{\pi} \int_0^\pi \sin^2 jx = 1 \right).$$

$$\lambda_j = +j^2 \to +\infty \quad \text{as } j \to +\infty.$$

Thm) $\{w_j\}$ spans $L^2(\Omega)$

Lemma) $\{w_j\}$ spans $H^1_0(\Omega)$ in L^2-norm.

i.e. given $f \in L^2(\Omega)$, we have

$$f = \sum_{i=1}^{\infty} \langle f, w_i \rangle_{L^2} w_i$$

proof) Let $a_i = \langle f, w_i \rangle$

$$\lambda_I < 0, \quad \lambda_{I+1} \geq 0.$$

Given $k \in \mathbb{N}$, we define

$$f_k = f - \sum_{i=1}^{k} a_i w_i.$$

Then, $\langle f_k, w_l \rangle_{L^2} = 0 \quad \forall\, l \leq k.$

$$\Rightarrow f_k \in X_k = \operatorname{span}\{w_1, \cdots, w_k\}^{\perp}$$

$$\Rightarrow \frac{\int |\nabla f_k|^2 - V f_k^2}{\int f_k^2} \geq \lambda_{k+1} \to +\infty$$

$$\int |\nabla f|^2 - Vf^2$$

$$= \int |\nabla f_k + \sum_{i=1}^{k} a_i \nabla w_i|^2 - V\Big(f_k + \sum a_i \nabla w_i\Big)^2$$

$$= \boxed{\int |\nabla f_k|^2 - V f_k^2} + \sum a_i^2 \underbrace{\int |\nabla w_i|^2 - V w_i^2}_{= \lambda_i \|w_i\|_{L^2}^2}$$

$$+ \underbrace{\boxed{2 \langle \nabla f_k, \sum_{i=1}^{k} a_i \nabla w_i \rangle_{L^2} - 2\int V f_k \sum_{i=1}^{k} a_i w_i}}_{= J_k}$$

$$\geq \lambda_{k+1} \|f_k\|_{L^2}^2 + \sum \lambda_i a_i^2 + J_k$$

$$\geq \lambda_{k+1} \|f_k\|_{L^2}^2 + \sum_{i=1}^{I} \lambda_i a_i^2 + J_k$$

(for all $k \geq I$)

Claim: $\langle \nabla f_k, \nabla w_\ell \rangle_{L^2} = \int V f_k w_\ell \quad \forall \ell \leq k$

i.e. $J_k = 0$.

So, if the claim is true

$$\int |\nabla f|^2 - V f^2 \, dx - \sum_{i=1}^{I} \lambda_i a_i^2$$

$$\geq \lambda_{K+1} \|f_K\|_{L^2}^2 \quad \forall K \geq I.$$

i.e. $\|f_K\|_{L^2} \leq \dfrac{C}{\sqrt{\lambda_{K+1}}} \to 0$ as $K \to +\infty$

i.e. $f = \sum_{i=1}^{\infty} \langle f, w_i \rangle w_i$

pf of claim) $l \leq K$

$$\langle \nabla f_K, \nabla w_l \rangle_{L^2} = \langle f_K, -\Delta w_l \rangle_{L^2}$$

$$= \langle f_K, V w_l + \lambda_l w_l \rangle_{L^2}$$

$$= \int V f_K w_l \, dx + \lambda_l \cancelto{0}{\langle f_K, w_l \rangle_{L^2}}$$

proof of Thm)

By functional analysis.

$C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$

and $C_0^\infty(\Omega) \subset H_0^1(\Omega)$.

$\Rightarrow H_0^1(\Omega)$ is dense in $L^2(\Omega)$.

Therefore. given $f \in L^2(\Omega)$

$\exists \{f_k\} \subseteq H_0^1(\Omega)$ s.t.

$$\lim_{k\to\infty} \|f_k - f\|_{L^2} = 0$$

We can apply the previous lemma to each f_k to show

$$f = \sum_{i=1}^{\infty} \langle f, w_i \rangle_{L^2} w_i \quad !!$$

We skipped the proofs of

1. $\boxed{\text{Regularity of weak solution}}$ ☆☆☆ important.

i.e. smoothness of w.

(c.f. Salsa, section 8, GT section 8)
(Evans " 6)

2. $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$